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Basic Probabilities

The probabilities that we'll be learning about build from the set theory that we learned last class, only this time, the sets are specifically sets of *outcomes*.

What are "outcomes"?

Roughly, ***outcomes*** are ways-things-can-go. So, if I'm tossing a coin, there are two ways-things-can-go: namely, it can land heads or it can land tails. The total list of possible ways-things-can-go with respect to a given occurrence is known as the ***outcome space***. In other words, the event space is equal to the *set of all ways-things-can-go*. So, in the case of a single coin toss, this would be equal to the set {Heads, Tails}. I like to visualise the outcome space as a table (the shaded area is the entire event space):

Toss 1

Heads	Tails
H	T

Fig.1

I emphasised 'single' above to raise the following point: outcomes are not just identical with their description (e.g. 'one heads, one tails')--it matters *how* the outcomes came about. To illustrate, imagine I'm performing two tosses of a single coin. It is tempting to think that the outcome space should look like this {(Heads, Heads), (Tails, Tails), (Heads, Tails)} (i.e. that there are only three possible ways-things-can-go).

2 Heads	2 Tails	Heads/Tails
(H, H)	(T, T)	(H, T)

Fig.2

After all, what difference does the *order* in which the results land matter--isn't it just the same if the coin lands heads first as if it lands tails first?

In a word, ***no***. *It's not at all the same, where probabilities are concerned!* What is important is that there are *two* different ways in which the (Heads, Tails) result can come about. This is why I

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called outcomes 'ways-things-can-go'--this includes how they get there. Thus, the outcome space for two coin tosses is accurately as follows: {(Heads, Heads), (Heads, Tails), (Tails, Heads), (Tails, Tails)}

		Toss 2	
		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig.3

Another way to visualise the outcome space that might help make the previous point clear is in terms of branching possibilities. Consider a single coin toss. In the following, the vertical axis marks the progression of time. Imagining yourself at a time prior to Toss, we have two possible results:

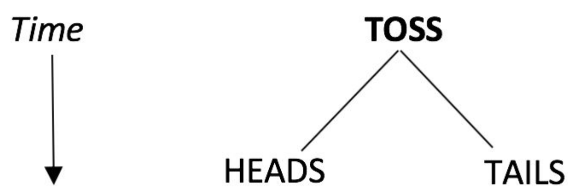

Fig. 4

Now, let's imagine ourselves at a time prior to Toss again, but this time take into consideration the second toss. We don't yet know how the first toss will go, so we can't eliminate either of those branches yet. We have to reason as follows: "supposing the first toss lands heads, the second could land either heads or tails; or, supposing the first toss lands tails, the second could land either heads or tails." Visually, we represent this as follows:

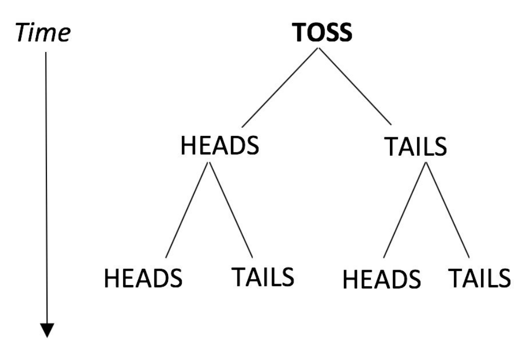

Fig. 5

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Using either the table or the branching chart can help to make calculating probabilities much easier. (NB: the branching chart will not be feasible for more cumbersome probabilities like “Monday Girl”.) In general, when we ask for the probability of some A here, we are asking for the ratio of the number outcomes where A comes about to the number of outcomes in the outcome space; i.e.

$$\frac{\text{\# of outcomes where A occurs}}{\text{\# of outcomes in outcome space}}$$

OR

$$\frac{\text{\# of ways A can come about}}{\text{\# of ways-things-can-go}}$$

Fig. 6

So, for instance, in the case of the coin tossed twice, **the probability of a coin’s landing heads at least once** can be calculated by looking at all of the outcomes in our outcome space, and tallying up the number of instances in which there is at least one heads. Recall Fig. 3, where green represented the total outcome space:

		Toss 2	
		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig.3

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Now, consider Fig. 7, where orange represents all of the outcomes with which we're concerned (i.e. all those that satisfy "heads lands at least once"). NB: In the course notes, the set of outcomes in orange is referred to as the ***event*** 'at least one heads':

		Toss 2	
		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig.7

So, the probability of heads landing at least once is going to be equal to:

$$\frac{\text{\# of orange squares (i.e. \# of outcomes satisfying the description)}}{\text{\# of green squares (i.e. \# of outcomes in the outcome space)}}$$

which equals…

$$\frac{3}{4}$$

Conditional Probabilities

Conditional probabilities are really no different to basic probabilities, for our purposes; the only thing that changes is the reference class, or in more practical terms, the denominator of our fractions from above. In these cases, instead of asking about the probability of A given *all* the ways-things-can-go, we are concerned about the probability of A given *some (subset) of* the ways-things-can-go. In practice, this means counting a different set of outcomes (the members of a different event) for our denominator (usually one that is a proper subset of, or some selection of, the outcome space).

So, imagine we want to know the **probability of heads landing at least once given that tails lands at least once**. We begin, as before, by returning to Fig. 3:

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Toss 2

		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig.3

But this time, we need to check the number of outcomes that satisfy the “given that” clause first! In this example, we want to know the probability of something ***given that tails lands at least once***. So let’s find all of the outcomes in which tails lands at least once:

Toss 2

		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig.8

I’ve shaded these boxes in a lighter green deliberately. If you like, with respect to the conditional probability we are trying to calculate, this is our new, restricted outcome space. We’re now really only concerned with the following:

Toss 2

		Heads	Tails
Toss 1	Heads	(H, H)	**(H, T)**
	Tails	**(T, H)**	**(T, T)**

Fig. 8.1

Given this new restricted ‘given that’ space, we now want to know how many of *these* outcomes satisfy the description “**heads lands at least once**”. And this time, our answer is **2** (rather than 3, as in the first example).

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Toss 2

		Heads	Tails
Toss 1	Heads	(H, H)	(H, T)
	Tails	(T, H)	(T, T)

Fig. 9

So, the conditional probability of heads landing at least once given that tails lands at least once is equal to

$$\frac{\text{\# of orange squares (i.e. \# of outcomes satisfying the description)}}{\text{\# of green squares (i.e. \# of outcomes in the 'given that' space)}}$$

which equals…

$$\frac{2}{3}$$

Conditional Probabilities, Pt. 2 -- The Formula

The formula you were given for conditional probabilities looks like this:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

But why? Let's try to make sense of this using more diagrams.

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Imagine the following is our outcome space:

Fig. 10

When we ask for the probability of some A, we want to know the ratio of the red area below (i.e. the set of all A outcomes, or the A event) to the entire rectangle area (i.e. the set of all possible outcomes). This represents the ratio of the number of outcomes satisfying A to the number of outcomes in the outcome space:

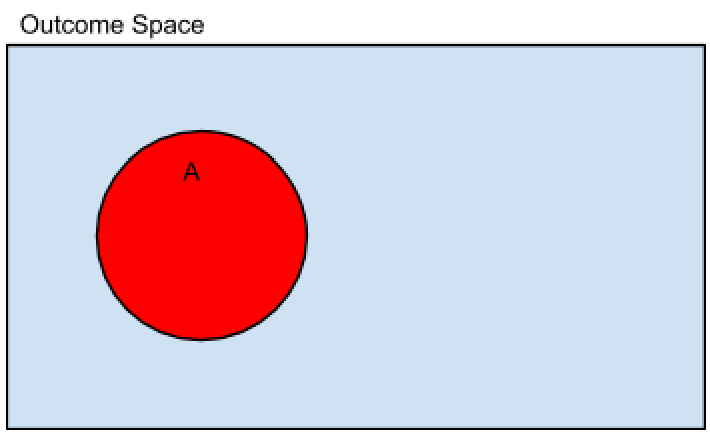

Fig. 11

Now let's supposed we're also concerned with another kind of outcome B:

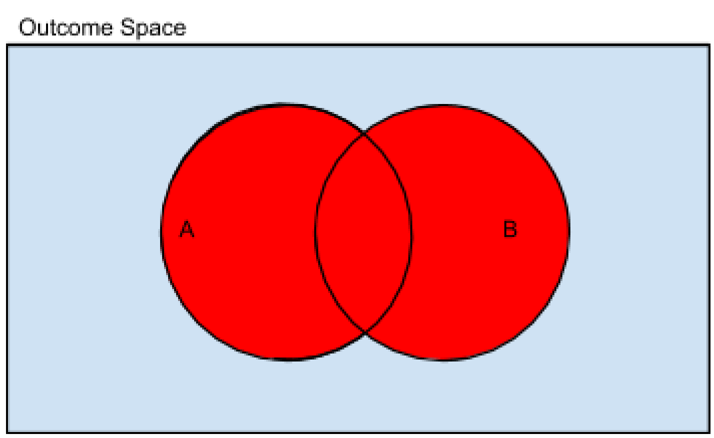

Fig. 12

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If each of the shaded areas represents a set of outcomes, then the area shaded red represents (A ∪ B), i.e. all the outcomes that are either A or B.

Remember what was said before about conditional probabilities and restricting the outcome space? Here's how we can represent that using venn diagrams. Suppose we are interested in the conditional probability of B given that A (i.e. Pr(B|A)). It was the 'given that' clause that constituted the restricted 'given that' space--that is to say, it was the 'given that' clause that described what figures as the denominator in our fraction (see p.6). Thus, the denominator in our formula for conditional probabilities is the probability of the 'given that' outcome--in this case A. Visually, then, the denominator is constituted by the area shaded in YELLOW below:

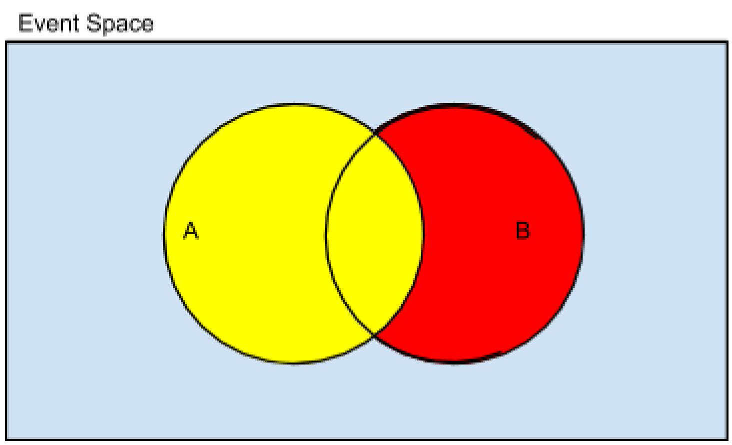

Fig. 13

Now, return to the original formula for conditional probabilities:

$$\Pr(B|A) = \frac{\Pr\,(A \cap B)}{\Pr\,(A)}$$

We've just figured out why the denominator is such as it is. Now let's figure out why the numerator consists of Pr(A ∩ B).

Remember, in a conditional probability, we wanted to know how many outcomes in the restricted 'given that' space (in this case, the A space in yellow) satisfy the description--in this case, B. In other words, we need first to figure out how many A's are also B; then we can

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calculate the ratio of the **# of A's-that-are-B** to the **# of A's**. In our venn diagrams, the shaded areas represent sets of outcomes. If we want to know the number of events that are A's **and** B's, then we want to know which outcomes fall into **both** sets. In other words, we are interested in the **intersection** of A and B. Hence **Pr(A ∩ B)**! Visually, we are concerned with the area in orange below:

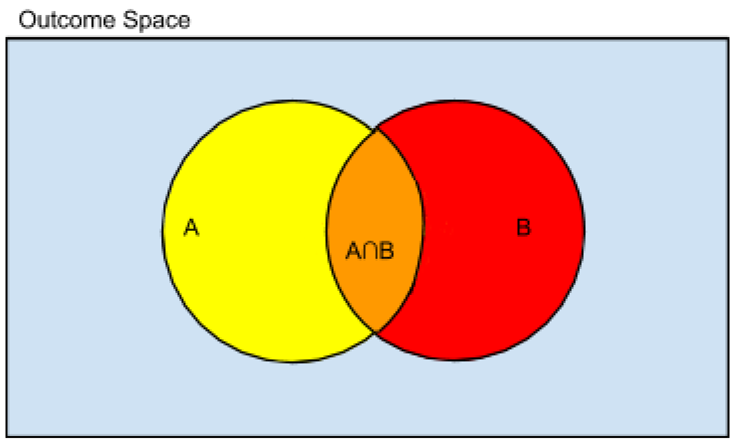

Fig. 14