A. Using the following symbolisation key:

domain: animals and stairs

Symbolise the following English sentences in FOL as best you can. Be warned that they are deliberately fiddly:

The key to this section lies mainly in correctly identifying the *scopes* of the quantifiers in each proposition. The paraphrase step will be hugely helpful in getting this right. In each paraphrase step, we'll determine (a) how many of each thing there is, and (b) what relation those things bear to one another.

1. A mouse is on a stair.

Recall from last worksheet that 'a' is to be paraphrased 'there exists a'. This is another way of saying 'there exists at one'. Importantly that's *all* 'a' tells us. Just that there's at least one of that thing--there may well be more of them! So, based on the original sentence, we know that:

- a. there is at least one mouse
- b. there is at least one stair

Great. That's how many of each thing there is. Now to work out the relation between them. In this case, we know of at least one mouse and at least one stair that that mouse is on that stair. So we can paraphrase the original sentence as follows:

There is at least one mouse, and there is at least one stair, and that mouse is on that stair.

The use of the word 'that' is important. It tells us that we are referring to the same mouse and the same stair in both instances. In other words, it indicates to us the scope of the relevant quantifiers.

Now we can formalise this 'chunk' at a time, like we did last week.

There is at least one mou	ise and	there is at least one stair,
хМхЕ	٨	ЗуSy
and	that mouse is on	that stair.
٨	Оху	/

Finally, we can put these together in a way such that <u>all of the variables are bound</u>! For instance, this would be wrong:

$\exists xMx \land \exists ySy \land Oxy$

Since the x and y in the last conjunct are not under the scope of any quantifier. To fix this, we need to use brackets like so:

$\exists x(Mx \land \exists y(Sy \land Oxy))$

2. The mouse is on a stair.

First, how many mice and how many stairs do we have? 'The mouse' tells us that there is *exactly one* (or, one and only one) mouse. 'A stair' tells us that there is at least one stair.

What relation do they stand in? Answer: The exactly one mouse is on that stair.

So our paraphrase is as follows:

There is exactly one mouse, and at least one stair, and that mouse is on that stair.

Now to formalise the definite description. There are two possible ways of doing this.

i. $\exists x(Mx \land \forall y(My \rightarrow y=x))$

This formalisation amounts to saying that there is at least one x that's a mouse (this is the first part), AND for all other things in the domain, IF it's a mouse, then it's identical to x. The result is that everything in the domain is either identical with x or isn't a mouse.

ii. $\exists x (\forall y (My \leftarrow \rightarrow y=x))$

This formalisation states that, there exists some x such that, for ALL y in the domain, y is a mouse if and only if y is identical to x. For a biconditional to be true, the antecedent and consequent have to be true together and false together. This biconditional says that the one and only situation in which y is a mouse is that in which y is identical with x.

We can use either one of these formalisation for the definite description. I'm going to stick with the first, but it really is a mere matter of preference.

We have what we need to formalise each of the pieces now:

There is exactly one mouseandat least one stair, $\exists x(Mx \land \forall y(My \rightarrow y=x))$ \land $\exists zSz$ andthat mouse is on that stair.

∧ Oxz

Now to put this together in a manner that respects scope.

$\exists x(Mx \land \forall y(My \rightarrow y=x) \land \exists z(Sz \land Oxz))$

3. A mouse is on the stair.

Same process as before. First, how many of each thing?

- at least one mouse
- exactly one stair

Relation between these? That mouse is on that stair.

Paraphrase:

There is at least one mouse, and there is exactly one stair, and that mouse is on that stair.

Formalise the parts:

There is at least one mous	se and	there is exactly one stair,
З×Мх	٨	$\exists y(Sy \land \forall z(Sz \rightarrow z=y))$
and	that mouse is o	n that stair.
٨	Oxy	/

Finally, put it together, minding the scope of the quantifiers.

$\exists x(Mx \land \exists y(Sy \land \forall z(Sz \rightarrow z=y) \land Oxy))$

Now, you might notice that this answer is slightly different than the one given in Tim's answer sheet. In general, there is going to be more than one correct way to formalise a bit of natural language. In this particular case, the difference between the two answers is really just a difference in the order of the conjunction. (NB: this has the result that a different quantifier is the main operator. In this particular case, this doesn't end up making a difference. However, as you'll see in a later worksheet, this will make a *big* difference when we've got two different (i.e. universal and existential) quantifiers in play.)

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4. The mouse is on the stair.

How many of each thing?

- exactly one mouse
- exactly one stair

Relation? That mouse is on that stair.

Paraphrase:

There is exactly one mouse and there is exactly one stair, and that mouse is on that stair.

Formalise the parts:

There is exactly one mous $\exists x(Mx \land \forall y(My \rightarrow y))$	se =x))	and A	there is exactly one stair $\exists z(Sz \land \forall y(Sy \rightarrow y=z))$
and ∧	that r	nouse is o <mark>Oxz</mark>	n that stair.

But hang on! How come I can use y twice here?? The reason is that the scopes don't clash. The scope of the first ' \forall y' ends when the brackets close. In other words, the second instance of 'y' doesn't fall under the scope of the first ' \forall y' quantifier. It has a quantifier of its own. I should note though, it is often easier, when you're working these problems out, just to use different variables for each quantifier. And, it would not be incorrect to use a different variable ('t' for instance) instead, like so:

There is exactly one mou	se and	there is exactly one stair
$\exists x(Mx \land \forall y(My \rightarrow y=x))$	^	$\exists z(Sz \land \forall t(St \rightarrow t=z))$
and tha	it mouse is	on that stair.
^	0	xz

I will often do this in my own work, since I won't always know beforehand what the scope of each of the quantifiers is going to be. In either case, just be mindful of scope.

Finally, put the pieces together:

 $\exists x(\mathsf{M}x \land \forall y(\mathsf{M}y \to y=x) \land \exists z(\mathsf{S}z \land \forall y(\mathsf{S}y \to y=z) \land \mathsf{O}xz))$

One final thing to note: in the answer key, Tim pulls the second existential quantifier out in front of the whole proposition. Once again, as long as all the pieces that should fall under the quantifier's scope do so, this is fine.

5. The mouse with clogs on is on the stair.

Questions 5 and 6 here are largely the same as the previous 4. The difference lies in the respective definite descriptions. We have some extra details added to our definite descriptions. So, let's first sort out what sorts of things we're talking about. To figure this out, we look to what comes after the quantifier indicator words. Here's our quantifier indicator words in BLUE:

The mouse with clogs on is on the stair.

And here's the scope over which those words range in RED:

So, how many of each thing have we got?

- exactly one mouse with clogs on
- exactly one stair

So, to paraphrase:

There is exactly one mouse with clogs on, and there is exactly one stair, and that mouse is on that stair.

Now let's sort out how to formalise 'there is exactly one mouse with clogs on'. To see how to do this, let's start with our formalisation of 'there is exactly one mouse':

$\exists x(Mx \land \forall y(My \rightarrow y=x))$

We know that this needs changing to reflect that there is exactly one mouse *with clogs on*. It is tempting to do this:

$\exists x(Mx \land \forall y(My \rightarrow y=x) \land Cx)$

But this would be wrong. Why? Because we've ruled out the possibility of there being any other mice! What we've said here is that there's exactly one mouse, and that mouse has clogs on. What we *wanted* to say was that there was *exactly one mouse with clogs on*. Importantly, the latter of these leaves open the possibility that there are all sorts of other mice about. It only rules

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out the possibility that any other mice *have clogs on*. To secure this, we need to say that for all y in the domain, if it's a mouse *with clogs on* THEN it's identical with x. Hence:

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\exists x(Mx \land Cx \land \forall y((My \land Cy) \rightarrow y=x))
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Now let's formalise the other bits:

There is exactly one mouse with clogs on $\exists x(Mx \land Cx \land \forall y((My \land Cy) \rightarrow y=x))$ and there is exactly one stair $\land \exists z(Sz \land \forall y(Sy \rightarrow y=z))$

and that mouse is on that stair. \bigwedge Oxz

And finally, put the parts together:

 $\exists x(Mx \land Cx \land \forall y((My \land Cy) \rightarrow y=x) \land \exists z(Sz \land \forall y(Sy \rightarrow y=z) \land Oxz))$

6. The mouse on the stair has clogs on.

Let's begin in the same way as we began the previous question: first, by identifying the quantifier indicator words, and second by identifying the scope of those indicator words.

First, indicator words in blue:

The mouse on the stair has clogs on.

Second, scope. Here's the scope of first 'the':

The mouse on the stair has clogs on. ↓_____

And here's the scope of the second 'the':

The mouse on the stair has clogs on. $|__\uparrow$

So, how many of each thing have we got?

- exactly one stair

- exactly one mouse on that stair

To paraphrase:

There is exactly one stair, and there is exactly one mouse on that stair, and that mouse has clogs on.

In the next step—the formalizing step—it's important for us to remember that the stair in the description of the mouse has to be the same stair picked out by the definite description. So, the description of the mouse has to fall within the scope of the quantifier that is quantifying over stairs.

Let's formalize the 'chunks' first:

There is exactly one stair $\exists x(Sx \land \forall y(Sy \rightarrow y=x))$	and ∧	there is exactly one mouse on that stair, $\exists z(Mz \land Ozx \land \forall y((My \land Oyx) \rightarrow y=z))$
and ∧	that mo	ouse has clogs on. Cz

Finally, put the parts together in a manner that respects scope:

$\exists x(Sx \land \forall y(Sy \rightarrow y=x) \land \exists z(Mz \land Ozx \land \forall y((My \land Oyx) \rightarrow y=z) \land Cz))$

It should now be clearer why the answers to 5 and 6 must be different. 5 is compatible with there being other mice on the stair, as long as there is one and only one mouse on that stair with clogs on. On the other hand 6 is NOT consistent with there being other mice on the stair—there can be one and only one mouse on the stair; but 6 *is* consistent with there being other mice in other places, with or without clogs on (just as long as they are not on the stair).

SECTION B - Show that the following arguments are invalid

2.
$$\forall x(Rx \to Dx), \forall x(Rx \to Fx) \therefore \exists x(Dx \land Fx)$$

To give interpretations, we don't always have to think of things from the actual world. If you are more comfortable doing it that way, that's fine, but be sure that there is no ambiguity to your interpretation. Also, make sure it would be clear to anyone what the extension of your

predicates is meant to be. So, for instance, making the domain the set of all of your friends will not help a marker or examiner know what you mean.

One way to work out interpretations is to go about it rather mechanically. In this case, I'll use numbers and a table:



Each column of the table is devoted to each predicate. I need to find an interpretation that shows that the argument in the question is invalid. SO, I need to give an interpretation that makes the premises true and the conclusion false.

I haven't set the domain yet. Let's suppose I add something to the domain -- 1 -- and say of it that it's **R**.

R	D	F
1		

Now, to ensure that the premises are true, it has to be the case that 1 is also **D** and **F**, since $\forall x(Rx \rightarrow Dx)$ and $\forall x(Rx \rightarrow Fx)$.

R	D	F
1	1	1

But now I've got something that is both **D** and **F**. In other words $\exists x(Dx \land Fx)$ comes out true. So that won't work as an interpretation showing the argument to be *invalid*. For that, I needed the conclusion to come out false!

In order for the conclusion to be false, there cannot exist anything that is both **D** and **F**. So, this time, I'll start by saying there is one thing in my domain -- 1 -- and I'll say of it that it's **F** but not **D**.

R	D	F
		1

Okay, now the conclusion is false. But are the premises true? Answer: Yes! It's true that, for *all* the things in the domain, IF it's **R**, THEN it's **D**, and IF it's **R**, THEN it's **F**. This is because, there is *nothing* in the domain that is **R**.

Here's how to give the interpretation properly, then:

Let the domain be the number 1. Let 'Fx' be true of 1, and 'Dx' and 'Rx' be false of 1.

I've now given an interpretation that shows the original argument to be false, and I've done so without having to search for a real-life example.

The tricky thing about interpretation-questions is that there is always going to be more than one correct answer. Also, there isn't a definite decision procedure we can provide that will get you to a correct answer every time. In other words, it will take some practice.

4. $Na \wedge Nb \wedge Nc$ $\therefore \forall xNx$

I'm going to do this one mechanically again. To make the premise true, we need at least one thing in the domain to be \mathbf{N} . To make the conclusion false, we need at least one thing in the domain NOT to be \mathbf{N} . So, let's imagine a domain with four entities:

3

4

Now let's decide which ones are called 'a', 'b', and 'c':

1

1	2	3	4
а	b	С	

2

We know that 'Nx' has to be true of the things named 'a', 'b', and 'c', so 'Nx' must be true of 1, 2, and 3, in our domain.

Clearly, we don't want to make 'Nx' true of 4 as well, because this would make it true that ALL the things in the domain are \mathbf{N} . So, our interpretation would read as follows:

Let the domain be the numbers 1, 2, 3, 4. Let 'a' name 1, 'b' name 2, and 'c' name 3. Let 'Nx' be true of 1, 2, and 3. Let 'Nx' be false of 4.

The reason why this answer is different from that in the answer key is that it is possible for variables to name the same element of the domain! So, 'a', 'b', and 'c' might all name the number 3, for instance, and this interpretation would still work.

6.
$$\exists x(Ex \land Fx), \exists xFx \to \exists xGx \therefore \exists x(Ex \land Gx)$$

I'll use the same process as above here. The easiest place to start will be by making the first premise true. So:



Now I need to make the second premise true, while still ensuring that the conclusion is false. To do this, note the scope of the quantifiers in the second premise! *It need not be the same object that is both* **F** *and* **G**. The scope of the first quantifier only ranges as far as the 'Fx'. In fact, if I *did* make the same object **F** and **G**, then the conclusion would be true (since that quantifier ranges over both conjuncts). So, I need some <u>distinct</u> object to be **G**. Hence:

E	F	G
1	1	
		2

Finally, the proper statement of my interpretation:

Let the domain be the numbers 1 and 2. Let 'Ex' and 'Fx' be true of 1, and false of 2. Let 'Gx' be true of 2 and false of 1.

7. $\forall xOxc, \forall xOcx \therefore \forall xOxx$

In this question we're dealing with two-place predicates -- i.e. relations. To represent these, I'm going to use arrows. Remember, though, that is something that we'll have to be explicit about in the statement of our interpretation. Let's start by figuring out what the diagram should look like before we worry about how to express the interpretation.

We've got one name 'c' in our premises, so we know we need at least one thing in our domain.

1

Let 'c' name 1.

Now let $x \rightarrow y$ represent 'Oxy'.

To make both premises true now, it needs to be the case that (1) everything stands in relation O to c, and (2) c stands in relation O to everything. That is to say, everything has to point to c, and c has to point to everything. But there's only one thing in the domain, so it has to point to itself, and that will satisfy both of the premises.

In this diagram, everything points to 1 (i.e. c) and 1 (i.e. c) points to everything. So the premises are true. BUT, the conclusion is true too! The conclusion states that everything stands in an **O**-relation to itself. And this is true on the interpretation we've just given.

So, how do we modify our interpretation to make the conclusion false? We know the first thing we'll need to do is add something to our domain. So let's add 2 to our domain.



2

Now let's make the premises true again. To do this, everything had to point to c (i.e. 1) and c (i.e. 1) had to point to everything. Hence:





Here, $\forall xOxc$ and $\forall xOcx$ are both true on an interpretation where 'c' names 1. What's more, now it's <u>false</u> that $\forall xOxx$ because 2 doesn't have an arrow pointing to itself. So we've figured out what the interpretation has to look like.

Here's what the final answer should look like:

Let the domain be the numbers 1 and 2. Let $x \rightarrow y$ represent Oxy. Finally, let the following diagram represent the relation Oxy in the specified domain:



9. $Lab \rightarrow \forall xLxb, \exists xLxb$. Lbb

Here the easiest place to start is with the existential premise (rather than with the conditional. Given the existential we know there has to be at least one thing in our domain.

1

For $\exists xLxb$ to be true, something needs to be b. So, let 'b' name 1. Let's also let ' $x \rightarrow y'$ represents *Lxy*. In this case, we might be tempted to do this:



And this would indeed make $\exists xLxb$ true. But, this also makes the conclusion true. And we want to give an interpretation on which the conclusion is false. So, the diagram just given is a bad place to start. Instead, let's suppose there's something else in the domain that points to b (i.e. 1).

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Now we've satisfied the second premise ($\exists xLxb$). 1 is named by 'b' and something points to 1.

We have a second name 'a' to deal with now. What should 'a' name? Given this latest diagram, if we let 'a' name 2, we'll render the first premise false. That's because, if 2 is named by 'a', then it will be true that *Lab* (since, as you'll recall, 1 is named by 'b'). But, it will be *false* that $\forall xLxb$, since not everything points to b (i.e. 1) -- in particular 1 (i.e. b) doesn't point to b! So the antecedent of the conditional is true, but the consequent is false; hence, the conditional is false (i.e. premise 1 is false). What's more, if we added this arrow (pointing from 1 to 1) in order to make the first premise true, this would also make the conclusion of the argument true, since it would make *Lbb* true. Therefore, 'a' cannot name 2.

So, let 'a' name 1! Now it's false that *Lab*. In this case, the antecedent of the conditional is false, making the conditional true. So the first premise is true. The second premise is also true because 2 points to 1 (i.e. b) so *something* is such that *Lxb*. And finally, the conclusion is false because it's not the case that *Lbb* (no arrow points from 1 to 1).