Worksheet 7

In this worksheet, we are working with set-theoretic notions. In this handout, I'll walk through the basic set-theoretic terminology. I'll then go through the questions from the worksheet.

SET

A set is just a collection of objects. These are *materially defined* -- that is to say, they are defined according to their *members*. BUT, they are <u>not</u> identical with their members. To use the common example, Socrates (the man) is <u>not</u> identical with the set containing Socrates (i.e. {Socrates}). However, the set {Socrates} <u>is</u> identical with {Socrates, Socrates, Socrates}, since they have exactly the same members.

EMPTY SET

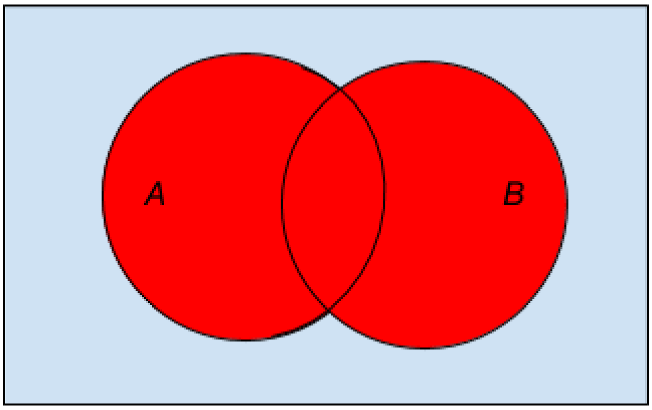
A set with no members. We represent the empty set either using the symbol '∅' or by using a pair of curly brackets like so: '{ }'.

AXIOM OF EXTENSIONALITY

This brings us to the axiom of extensionality. This axiom states that <u>*two sets A and B are identical if and only if, for all x, x is a member of A if and only if x is a member of B.*</u> In other words, sets are identical if and only if they have exactly the same members.

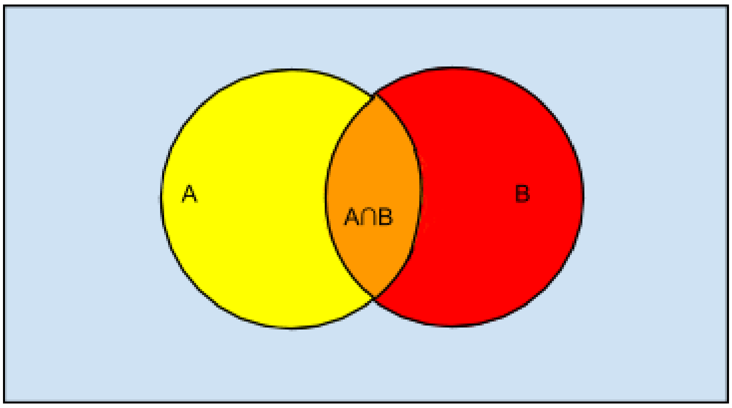
UNION

The union of two sets A and B ($A \cup B$) contains all x such that x is a member of A OR x is a member of B. Think of this in terms of the following diagram:

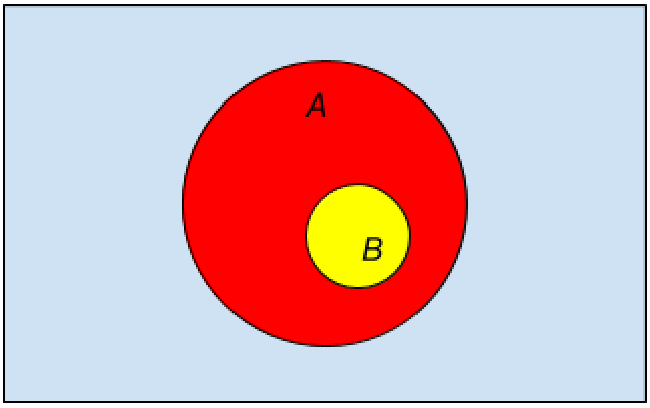


The union of A and B includes all of the objects in the red area: it includes the objects just in A, the objects just in B, and the objects in both A and B.

INTERSECTION
The intersection of two sets A and B (A∩B) contains all x such that x is a member of A AND x is a member of B. In other words, it is the set containing all of the members in the orange area below. The members in the area where the sets *intersect*.

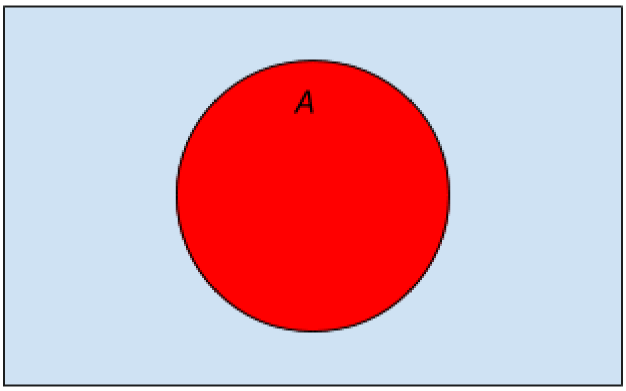


SUBSET
When we think of subsets, we typically think of one set being 'inside' another, like so:



In this case, we would say that B is a subset of A (i.e. B⊆A). In order for some B to be a subset of some set A, it must be true that, all of the members of B are also members of A. From the diagram, we can easily see that anything that's in the B-circle is also in the A-circle.

The important thing to notice about this definition is this: it follows that, for any set A, A⊆A. That is to say, every set is a subset of itself. This sounds awfully weird! But it follows from our definition because, it's always true of A that, all of the members of A are also members of A.

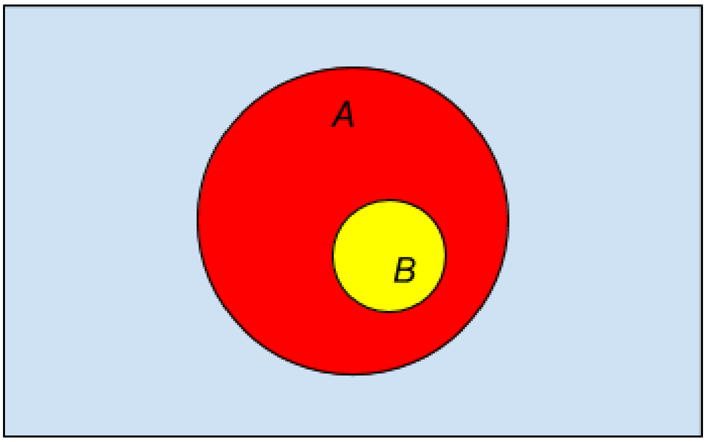


In fact, we can say of any two sets that if B⊆A and A⊆B, then A and B are identical.

The relation we use when we want to talk about proper containment is PROPER SUBSETHOOD.

PROPER SUBSET

B is a proper subset of A (i.e. $B \subset A$) just in case B is a subset of A, AND there is some member of A that is not a member of B. In other words, B is a proper subset of A if and only if A is NOT a subset of B. Have a look at this diagram again:



All of the things in the B-circle are in the A-circle. BUT, there are things in the A-circle that aren't in the B-circle. So B is a proper subset of A.

(Question to ponder: Could there ever be two sets A and B for which $B \subset A$ and $A \subset B$?)

POWER SET

The *power set* of a set S is the set containing all subsets of S. Here's a general process for generating a power set:

Imagine you have some set S = {1, 2, 3}. What are its subsets?

- { } ← The empty set is a subset of ALL sets. For A to be a subset of B, it has to be that, for all x, if x is a member of A then x is a member of B. But the empty set has no members, so the antecedent is false for all x. Therefore it is trivially true that the empty set is subset of any set.
- {1}
- {2}
- {3}
- {1, 2}
- {1, 3}
- {2, 3}
- {1, 2, 3} ← Every set is a subset of itself. It is always true that, for all x, if x is a member of A then x is a member of A. So A is always is subset of A.

So, $\mathscr{P}$(S) = {{ }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

In general, if a set S contains *n* members, then its powerset $\mathscr{P}$(S) contains 2^n members. In our example, S contained 3 members, and $\mathscr{P}$(S) contains 2^3=8 members.

1. **Let `C' and `D' name the sets of all cats and dogs, respectively. Let `Rxy' express the relation x is older than y and let `f' name Fido. Use set-theoretic notation to write down expressions for the following sets:**

    (a) **The set of all animals who are cats or dogs.**
    In general, when you see 'or' you can take that as an indication that you are after a union between sets (since the union of A and B is the set whose members are either in A or in B).

    Here, we want the set whose members are either a cat or a dog. Hence:

    $$\mathbf{C \cup D}$$

    (b) **The set of all cats younger than Fido.**
    Okay, here let's take things in steps. We can divide the description into two parts: (i) cats (ii) younger than Fido. First, let's pick out all of the cats. We already have a set defined for this:

    $$\mathrm{C}$$

    Now let's figure out how to say of *x* that *x* is younger than Fido (this is just what we did in FOL):

    $$Rfx$$

    Okay, now we know we want the set of *cats* such that they are younger than Fido, so we specifically want the *x*s that are in C and for which *Rfx* is true. Hence:

    $$\boldsymbol{\{x : x \in \mathrm{C} \land Rfx\}}$$

    (c) **The set of all sets of dogs younger than Fido.**
    Here again we'll take things in steps. First, let's define 'the set of all dogs younger than Fido'. This is going to look exactly like the set of all cats younger than Fido, except we'll swap the 'C' out for a 'D':

    $$\{\mathrm{x : x \in D \land Rfx}\}$$

    Now, the question asks for the set of all sets of that description. This means we need the powerset of this set! Hence:

    $$\boldsymbol{\wp\{x : x \in \mathrm{D} \land Rfx\}}$$

    Finally, recall that powersets always include the *empty set*. But the empty set is not a set of dogs. So to 'subtract' the empty set from the members of the above, we want the complement of this and the *set containing the empty set*.

    $$\boldsymbol{\wp\{x : x \in \mathrm{D} \land Rfx\} - \{\varnothing\}}$$

(d) The set of all dogs younger than every cat.

First, let's try to figure out how to pick out the set of all *x*s younger than every cat. Here we want all x's such that, for all *y*, if *y* is a cat, *y* is older than *x*:

$$\{x : \forall y(y \in C \rightarrow Ryx\}$$

Now, that is the set of ALL things younger than all cats. We only want the x's that are dogs. In other words, we want the set of things that are dogs <u>AND</u> they are younger than all cats. The 'and' tells us we want the <u>intersection</u> between the set of dogs and the set of things younger than all cats:

$$\mathbf{D \cap \{\boldsymbol{x} : \forall \boldsymbol{y}(\boldsymbol{y} \in C \rightarrow \boldsymbol{Ryx}\}}$$

(e) The set of all possible mixed pairs of cats and dogs involving Fido.

Now, the question does not specify this, but I am going to assume that Fido is a dog. (**NB:** In general, if you are assuming something that is consistent with the question but not specified in the question, then you should make this explicit above your answer.)

Given that Fido is a dog, then all the mixed pairs involving Fido will be pairs where one part of the pair is Fido, and the other part of the pair is a cat.

Since we are told that we want <u>pairs</u>, we know that we want the <u>product</u> of two sets. So now the question is: which sets? We know we want the set of all cats to be one of them, because the question asks for ALL mixed pairs (and if Fido is a dog, then all the mixed pairs will be pairs with some cat, for each cat). So, you might be tempted to do this:

$$C \text{ X } f$$

The reason why this is wrong is that *f* is not a set! *f*, recall, is the name for Fido -- a particular object. But the <u>product</u> relation only holds between at least <u>two *sets*</u>. So we need another set on the other side of the product relation. Specifically, the set containing Fido. I.e. $\{ f \}$ Hence, our final answer should look like the following:

$$\mathbf{C \text{ X } \{ \boldsymbol{f} \}}$$

2. **Explain why it follows from Extensionality that there is at most one empty set.**
   The best way to answer a question like this is to begin with the definitions of the relevant terms.

   Extensionality tells us that two sets are identical iff they have exactly the same members.

   The **empty set** is the set containing no members.

   Now, take any two sets A and B that are empty sets. It follows from the definition of the empty set that A has no members, and that B has no members.

   By Extentionality, A is the same set as B iff, for all *x*, *x* is a member of A iff *x* is a member of B. This is vacuously true since there are no *x*'s such that *x* is a member of A or *x* is a member of B. Therefore, A is identical to B. But A and B were arbitrary, so this will be true of all empty sets. Therefore all empty sets are identical. Therefore there is only one empty set.

3. **Provide brief arguments for each of the following claims:**

   (a) A ∩ B ⊆ A ∪ (B ∪ C)
   You can give these arguments symbolically, as Owen does in his answer key. However, you can also do this in prose. I'll do the latter here so you can see what that would look like (it may also help make the formal argument clearer).

   Again, the best thing to do is to define your terms and then to operate from those definitions.

   On the definition of an intersection, the set A ∩ B is the set containing all *x* such that *x* is a member of A and *x* is a member of B.

   On the definition of a union, the set A ∪ (B ∪ C) is the set containing all *x* such that *x* is a member of A or *x* is a member of B or *x* is a member of C.

   If *x* is a member of A and *x* is a member of B, it follows that *x* is a member of A. And if *x* is a member of A, then it follows that *x* is a member of A or *x* is a member of B or *x* is a member of C (by the logic of disjunction). Therefore, all the members of A ∩ B are members of A ∪ (B ∪ C). Therefore A ∩ B is a subset of A ∪ (B ∪ C), i.e. A ∩ B ⊆ A ∪ (B ∪ C).

(b) A ∪ (B ∩ A) = A

To show that two sets are equivalent, you have to show that each of them is a subset of the other.

By the definition of union, *x* is a member of A ∪ (B ∩ A) iff *x* is either a member of A or *x* is a member of (B ∩ A). And by the definition of intersection, *x* is a member of (B ∩ A) iff *x* is a member of A and a member of B. So, in either case, *x* is a member of A. Therefore, A ∪ (B ∩ A) is a subset of A (by the definition of subsethood).

If *x* is a member of A, then *x* is a member of A ∪ (B ∩ A), since it suffice for *x*'s being a member of A ∪ (B ∩ A) that *x* is a member of A (by the definition of union). Therefor, A is a subset of A ∪ (B ∩ A).

Therefore, for all *x*, *x* is a member of A iff *x* is a member of A ∪ (B ∩ A). Therefore, by Extensionality, A ∪ (B ∩ A) = A.

(c) A ∩ (B ∪ ∅) ⊆ B ∪ C

x is a member of A ∩ (B ∪ ∅) iff *x* is a member of A and *x* is a member of (B ∪ ∅). *x* is a member of (B ∪ ∅) iff *x* is a member of B or *x* is a member of ∅. But ∅ is the empty set, so it has no members. Therefore, if *x* is a member of (B ∪ ∅), then *x* must be a member of B.

By the definition of union, if *x* is a member of B, then *x* is a member of B ∪ C.

Therefore, if *x* is a member of A ∩ (B ∪ ∅) then *x* is a member of B ∪ C. Therefore, by the definition of subsethood, A ∩ (B ∪ ∅) ⊆ B ∪ C.