S. Siriwardena
ss2032@cam.ac.uk

Worksheet 8
Solution Walk-Through (Further Exercises)

1. **Provide examples of relations that are:**
    a. **Reflexive and transitive but not symmetric**
    b. **Euclidean and transitive but not reflexive**
    c. **Symmetric and transitive but not reflexive**
    d. **Reflexive and symmetric but neither transitive nor Euclidean**

To begin answering this question, it is useful to rehearse the formal definitions of reflexivity, transitivity, symmetry and Euclideanness.

A relation is **reflexive** iff **$\forall x Rxx$**.

A relation is **symmetric** iff **$\forall x \forall y(Rxy \rightarrow Ryx)$**

A relation is **transitive** iff **$\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$**

A relation is **Euclidean** iff **$\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow Ryz)$**

Now, notice that the bottom three of these is defined *conditionally*. Since a conditional is always true whenever the antecedent is false, it can sometimes be the case that a relation is **vacuously** symmetric, transitive, or Euclidean. This will happen whenever nothing in the domain satisfies the antecedent of the relevant conditional. Relatedly, it need not be the case that *everything* in the domain stand in some relation in order for a relation to be symmetric, transitive, or Euclidean.

A second thing to note is that even though *x*, *y*, *z* are distinct *variables*, they all range over the *same objects*! These variables are in the scope of universal quantifiers. So, when you're thinking about examples of relations, don't fall into the trap of only assigning *x*, *y*, and *z* to distinct objects in your domain.

One thing that can help to make sure you've considered all of the permutations with respect to the variables and the objects in your domain is to give the objects in your domain names using logical constants (e.g. *a*, *b*, *c*). This may help to remind you that, for all of the variables, you should be able to substitute *any combination* of constants and have the proposition come out true. Another method than can help (when you're working out examples of relations) is to draw up a kind of matrix of all of the options. I'll demonstrate below.

S. Siriwardena
ss2032@cam.ac.uk

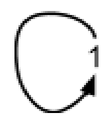
a. Reflexive and transitive but not symmetric

To answer this question, I might begin by using numbers as objects in my domain, and by giving each number a distinct name. (**NB:** You <u>must</u> specify the domain on which your relation is defined.)

As a rule of thumb, start with the smallest number of objects in your domain, and smallest number of arrows you can and then only add as many as you need to satisfy the relation specified.

Let's say I started with just one object in the domain. Let 1 be that object, and let 'a' name 1.

Now, to make this reflexive, I need to draw an arrow from 1 to itself:



Next we check to see if this is transitive by determining whether $\forall x \forall y \forall z((Rxy \land Ryz) \rightarrow Rxz)$ is true on this domain. In this case, there is only one object over which *x*, *y*, and *z* range -- in particular, over the object named 'a' (i.e. 1). Thus, we need only ask if the following is true: $((Raa \land Raa) \rightarrow Raa)$. This, is indeed true on the domain defined, so the relation is transitive.

Next we check to see if it is symmetric by determining whether $\forall x \forall y(Rxy \rightarrow Ryx)$ is true on this domain. Of course, one again, there is only one object in the domain, so we need only ask if $(Raa \rightarrow Raa)$ is true. This is true on this domain, so the relation is symmetric. **But we wanted a relation that was not symmetric**. So we need to add another object to the domain! Let the new object be 2, and let 'b' name 2.

Now, we go through the same steps as above. First we want to make it reflexive. Since we've added an object to the domain, we need to make sure that object has an arrow pointing to it.

We also want to make sure that this relation is transitive but not symmetric. Right now, both $\forall x \forall y \forall z((Rxy \land Ryz) \rightarrow Rxz)$ and $\forall x \forall y(Rxy \rightarrow Ryx)$ are true. Consider:

S. Siriwardena
ss2032@cam.ac.uk

x	*y*	*z*
a	a	a
a	a	b
a	b	a
a	b	b
b	a	a
b	a	b
b	b	a
b	b	b

These are all the possible combinations of *a* and *b* that the variables can range over. For each of these lines, the relevant conditionals are true:

x	*y*	*z*	**(Rxy∧Ryz)→Rxz**	**T?**	**(Rxy→Ryx)**	**T?**
a	a	a	(Raa∧Raa)→Raa	T	(Raa→Raa)	T
a	a	b	(Raa∧Rab)→Rab	T	(Raa→Raa)	T
a	b	a	(Rab∧Rba)→Raa	T	(Rab→Rba)	T
a	b	b	(Rab∧Rbb)→Rab	T	(Rab→Rba)	T
b	a	a	(Rba∧Raa)→Rba	T	(Rba→Rab)	T
b	a	b	(Rba∧Rab)→Rbb	T	(Rba→Rab)	T
b	b	a	(Rbb∧Rba)→Rba	T	(Rbb→Rbb)	T
b	b	b	(Rbb∧Rbb)→Rbb	T	(Rbb→Rbb)	T

So, looking at the table, we want to make an addition that keeps the green column true, but that results in some false line in the orange column. Thus, we need a uni-directional arrow between the objects in the domain. E.g.

S. Siriwardena
ss2032@cam.ac.uk

This will change the table in the following way:

x	*y*	*z*	**(Rxy∧Ryz)→Rxz**	**T?**	**(Rxy→Ryx)**	**T?**
a	a	a	(Raa∧Raa)→Raa	T	(Raa→Raa)	T
a	a	b	(Raa∧Rab)→Rab	T	(Raa→Raa)	T
a	b	a	(Rab∧Rba)→Raa	T	**(Rab→Rba)**	**F**
a	b	b	(Rab∧Rbb)→Rab	T	**(Rab→Rba)**	**F**
b	a	a	(Rba∧Raa)→Rba	T	(Rba→Rab)	T
b	a	b	(Rba∧Rab)→Rbb	T	(Rba→Rab)	T
b	b	a	(Rbb∧Rba)→Rba	T	(Rbb→Rbb)	T
b	b	b	(Rbb∧Rbb)→Rbb	T	(Rbb→Rbb)	T

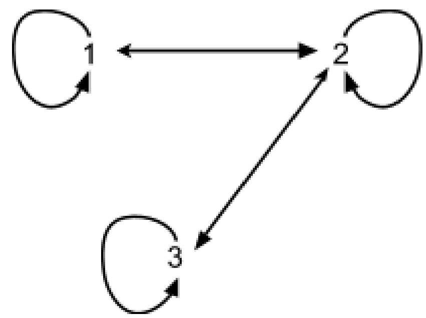
b. Euclidean and transitive but not reflexive

I won't run through the details of each of the rest of the questions. But I will provide model answers.

c. Symmetric and transitive but not reflexive

d. Reflexive and symmetric but neither transitive nor Euclidean

S. Siriwardena
ss2032@cam.ac.uk



2. **Two cards are drawn at random and without replacement from a standard pack of 52 cards. What is the probability that:**

(a) Both are aces?
Since, we are drawing two cards from the deck *without replacement*. So, when we draw the FIRST card, there are **52** in the deck, and when we draw the SECOND card, there are only **51** in the deck. Thus, there are 52 members in the set of possible first-cards, and 51 members in the set of possible second-cards.

Given this, how do we get the set of all possible *pairs* of first-cards and second-cards? Answer: The cartesian product!

Thus, the **outcome space = (Set of first-cards)X(Set of second-cards) = 52 x 51**

Now, in a full deck there are **4** aces. But, on the second draw, if the first draw was an ace, then there will only be **3** left on the second draw. Thus, there are **4x3** ways to draw two aces.

Therefore, the probability of drawing two aces is equal to **(4x3)/(52x51)**, i.e. **1/221**

(b) At least one is an ace?
Here again, we are interested in both draws, so the outcome space = 52x51.

Now there are **three** different ways of drawing at least one ace:
(i) the first is an ace, but not the second
(ii) the second is an ace, but not the first
(iii) the first and the second are aces

Taking each of these in turn:

(i) There are 4 aces, so there are **4x48** ways to draw an ace first but not second
(ii) Just as above, there are **48x4** ways to draw an ace second but not first
(iii) As in the last question, there are **4x3** ways to draw aces first and second

S. Siriwardena
ss2032@cam.ac.uk

Thus, the probability of drawing at least one ace =
((4x48)+(48x4)+(4x3))/(52x51) = **33/221**

(c) The second is an ace given that the first is an ace?
To answer this one, we need the formula for conditional probability to hand.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

In this case, we want Pr(2A|1A), thus we need:

$$\frac{Pr(2A \cap 1A)}{Pr(1A)}$$

Now, the intersection of the set of outcomes where two aces are drawn and the set of outcomes where one ace is drawn is just going to be identical to the set where both are aces. And in (a) we saw that the probability of both being aces equals **12/(52x51)**.

Second, the probability of the first draw being an ace = **(4x51)/(52/51)**

Finally, to divide fractions, invert and multiply! So Pr(2A|1A) =
[12*(52/51)]/[(4x51)*(52x51)]

And since (52/51) appears in both the numerator and denominator of that fraction, both instances cancel out. Thus, Pr(2A|1A) = **12/(4x51)** = **1/17**

(d) The first is an ace given that the second is an ace?
This answer will just be the same as the previous, as the reasoning will be exactly the same, except with ‘first’ and ‘second’ switched around. Thus Pr(1A|2A) = **1/17**.

(e) The first is a king or an ace?
Since all of the cards are in the deck, the **outcome space = {cards in a complete deck} = 52**.

S. Siriwardena
ss2032@cam.ac.uk

Since there are 4 kings in a deck and 4 aces in a deck, so this set of outcomes has **8** members.

Therefore, the probability of drawing a king or an ace is **8/52**, i.e. **2/13**.

(f) They are both hearts given that one is a heart?
We are here concerned with:

$$\Pr(\text{Pair with Two Hearts}|\text{Pair with One Heart})$$

(for short…)

$$\Pr(\text{2Hearts}|\text{1Heart})$$

Recall that we can calculate conditional probabilities according to the following formula:

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

So, using this, let's substitute the appropriate values from the card question into our formula:

$$\Pr(\text{2Hearts}|\text{1Heart}) = \frac{\Pr(\text{1Heart} \cap \text{2Heart})}{\Pr(\text{1Heart})}$$

Now we need to figure out the different ways that 2Hearts can come about and the different ways that 1Heart can come about.

The **outcome space = (Set of first-cards)X(Set of second-cards) = 52 x 51**

Okay. Now that we have our outcome space, let's figure out all the ways that we can draw 1 heart. We could…

S. Siriwardena
ss2032@cam.ac.uk

(i) First draw a heart, then draw something else
(ii) First draw something else, then draw a heart
(iii) First draw a heart, then draw another heart ***

IMPORTANT: drawing two hearts is a way to draw 1 heart! So (c) has to be included in the total number of ways that 1Heart can come about.

How many different ways can each of (a) through (c) be realised?

Well… there are **13 hearts** in total, and **39 non-hearts**. With this in mind, let's return to (i)-(iii):

(iii) First draw a heart, then draw something else = **13x39**

(ii) First draw something else, then draw a heart = **39x13**

(iii) First draw a heart, then draw another heart = **13x12**

Thus, 1Heart can come about the following number of ways:

(i) First draw a heart, then draw something else: **13x39**
(ii) First draw something else, then draw a heart: **39x13**
(iii) First draw a heart, then draw another heart: **13x12**

which equals…

13x(39+39+12)

Therefore, the probability of drawing one heart equals (i.e. Pr(1Heart))…

$$\text{Pr(1Heart)} = \frac{\text{\# of pairs with 1 heart}}{\text{\# of pairs}}$$

which equals…

$$\text{Pr(1Heart)} = \frac{13\text{x}(39+39+12)}{52\text{x}51}$$

S. Siriwardena
ss2032@cam.ac.uk

Now let's go through the same process for Pr(2Hearts).

How many different ways can 2Hearts come about? This answer is much simpler. It's just (iii) from above. I.e.

(iii) First draw a heart, then draw another heart: **13x12**

Thus, Pr(2Hearts) is equal to…

$$\text{Pr(2Hearts)} = \frac{\text{\# of pairs with 2 hearts}}{\text{\# of pairs}}$$

which equals…

$$\text{Pr(2Hearts)} = \frac{13\text{x}12}{52\text{x}51}$$

Okay, let's return to the equation for conditional probability that we started with:

$$\text{Pr(2Hearts|1Heart)} = \frac{\text{Pr(1Heart} \cap \text{2Heart)}}{\text{Pr(1Heart)}}$$

We've figured out the denominator now, i.e. Pr(1Heart). All that's left is to figure out the value of the numerator.

How many different events fall into the intersection of the set of 1Heart-events and the set of 2Heart-events? To answer this, let's return to the ways that 1Heart and 2Hearts can come about:

1Heart:

(i) First draw a heart, then draw something else: **13x39**
(ii) First draw something else, then draw a heart: **39x13**

S. Siriwardena
ss2032@cam.ac.uk

(iii) First draw a heart, then draw another heart: **13x12**

2Hearts:

(iii) First draw a heart, then draw another heart: **13x12**

Notice that the set of 2Hearts-events is a **proper subset** of the set of 1Heart-events. That is, all the ways that 2Hearts can come about are ways that 1Heart can come about. So, the set of events in the intersection of 1Heart and 2Hearts will be equal to the set of 2Hearts-events.

I.e.

$$\Pr(1\text{Heart} \cap 2\text{Hearts}) = \Pr(2\text{Hearts})$$

Given this, we can now say that…

$$\Pr(2\text{Hearts}|1\text{Heart}) = \frac{\Pr(2\text{Hearts})}{\Pr(1\text{Heart})}$$

And, since we've already calculated the probabilities for the numerator and the denominator here, all that's left is to substitute the respective values into the equation and do the arithmetic.

$$\Pr(2\text{Hearts}|1\text{Heart}) = \frac{\Pr(2\text{Hearts})}{\Pr(1\text{Heart})}$$

Substituting our values into the fraction…

$$\Pr(2\text{Hearts}|1\text{Heart}) = \frac{(13x12)/(52x51)}{(13x(39+39+12))/(52x51)}$$

S. Siriwardena
ss2032@cam.ac.uk

Which is equal to: **2/15**

(g) They are both hearts given that on is the queen of hearts?
Once again we are dealing with a conditional probability. This time of the following:

$$\Pr(\text{Pair with Two Hearts}|\text{Pair with Queen of Hearts})$$

(for short…)

$$\Pr(\text{2Hearts}|\text{Q})$$

Using our equation for conditional probabilities again, we get:

$$\Pr(\text{2Hearts}|\text{Q}) = \frac{\Pr(\text{Q} \cap \text{2Heart})}{\Pr(\text{Q})}$$

How many different ways can Q come about? That is, how many different ways can we draw two cards such that one of them is the queen of hearts?

(i) First draw queen of hearts, second draw any other card = **1x51**
(ii) First draw any other card, second draw queen of hearts = **51x1**

Thus, Pr(Q), i.e. the denominator, is equal to: **(51+51)/(52x51)**

Now let's turn to the numerator. How many outcomes (i.e. pairs of cards) are in the intersection of the set of pairs-with-queen-of-hearts and the set of pairs-with-2-hearts?

We know that, since we're dealing with an intersection, all of the pairs in the intersection will have the queen of hearts. We also know that the other card has to be *any other heart* <u>even if it is drawn before the queen</u>. Since there's only 1 way to be the queen of hearts, and 12 ways to be any other heart, there are the following ways of being an outcome in the intersection of the set of pairs-with-queen-of-hearts and the set of pairs-with-2-hearts:

(a) First draw queen of hearts, second draw any other heart = **1x12**
(b) First draw any other heart, second draw queen of hearts = **12x1**

S. Siriwardena
ss2032@cam.ac.uk

Thus, Pr(Q ∩ 2Hearts), i.e. the numerator, is equal to: **(12+12)/(52x51)**

And so, once again, all that remains is a bit of arithmetic now…

$$\Pr(2Hearts|Q) = \frac{\Pr(Q \cap 2Heart)}{\Pr(Q)}$$

Which is equal to: **4/17**

3. You stop two random parents on the street.

(a) Jane has two children. You ask her: 'Is at least one of them a girl?'--Yes. What is the probability that she has two girls?

Let's start by working out the outcome space. Jane had a first child and then a second child, so the outcome space would look like the following:

		First Child	
		Boy	Girl
Second Child	Boy	(B, B)	(B, G)
	Girl	(G, B)	(G, G)

We want to know the **Pr(2 girls|1 girl)**.

So, let's look just at the outcomes in our outcome space that are pairs including 1 girl:

		First Child	
		Boy	Girl
Second Child	Boy	(B, B)	(B, G)
	Girl	(G, B)	(G, G)

There are **3** outcomes such that 1 is a girl.

How many of *these* events are such that both are girls? Answer: **1**

First Child	
Boy	Girl

S. Siriwardena
ss2032@cam.ac.uk

Second Child	Boy	(B, B)	(B, G)
	Girl	(G, B)	(G, G)

So, Pr(2 girls|1 girl) = **1/3**

(b) Ian has two children. You ask him: 'Is at least one of them a girl born on a Monday?'--Yes. What is the probability that he has two girls?
The key to this question is getting the outcome space right. Let's start with the outcome space from the previous question…

		First Child	
		Boy	Girl
Second Child	Boy	(B, B)	(B, G)
	Girl	(G, B)	(G, G)

What this question calls for is a massive subdivision of this outcome space. We are adding to the possible ways of having two children, if you like… Take just the bottom-right corner, and imagine "zooming in" There are now **7x7** ways to have two girls, rather than just 1!

		First Child	
		Boy	Girl
Second Child	Boy	(B, B)	(B, G)
	Girl	(G, B)	**(G, G)**

					GIRL			
		Mo	Tu	We	Th	Fr	Sa	Su
	Mo							
	Tu							
	We							
GIRL	Th							

S. Siriwardena
ss2032@cam.ac.uk

Fr							
Sa							
Su							

So, doing this with the entire outcome space, we have the following:

		BOY							GIRL						
		Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
B	Mo	BM, BM	BM, BT	BM, BW	BM, BT	BM, BF	BM, BS	BM, BS	BM, GM	BM, GT	BM, GW	BM, GT	BM, GF	BM, GS	BM, GS
B	Tu	BT, BM	...												
B	We	BW, BM		...											
B	Th	BT, BM			...										
B	Fr	BF, BM				...									
B	Sa	BS, BM					...								
B	Su	BS, BM						...							
G	Mo	GM, BM							...						
G	Tu	GT, BM								...					
G	We	GW, BM									...				
G	Th	GT, BM										...			
G	Fr	GF, BM											...		
G	Sa	GS, BM												...	
G	Su	GS, BM													...

S. Siriwardena
ss2032@cam.ac.uk

In the previous question, we wanted to know the probability of an outcome being in the DARK BLUE area (below) given that it is in ANY BLUE area, and as the diagram shows, this is clearly 1/3.

		BOY							GIRL						
		Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
B	Mo	BM, BM	BM, BT	BM, BW	BM, BT	BM, BF	BM, BS	BM, BS	BM, GM	BM, GT	BM, GW	BM, GT	BM, GF	BM, GS	BM, GS
B	Tu	BT, BM	...												
B	We	BW, BM		...											
B	Th	BT, BM			...										
B	Fr	BF, BM				...									
B	Sa	BS, BM					...								
B	Su	BS, BM						...							
G	Mo	GM, BM							...						
G	Tu	GT, BM								...					
G	We	GW, BM									...				
G	Th	GT, BM										...			
G	Fr	GF, BM											...		

S. Siriwardena
ss2032@cam.ac.uk

G	Sa	GS, BM												...	
G	Su	GS, BM													...

This time, the light-blue area is much different. That is to say, *our restricted given-that set of outcomes is the set of pairs of children <u>such that one child was a girl born on a monday</u>*

Thus, we are concerned with the following outcomes:

		BOY							GIRL						
		Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
B	Mo	BM, BM	BM, BT	BM, BW	BM, BT	BM, BF	BM, BS	BM, BS	BM, GM	BM, GT	BM, GW	BM, GT	BM, GF	BM, GS	BM, GS
B	Tu	BT, BM	...												
B	We	BW, BM		...											
B	Th	BT, BM			...										
B	Fr	BF, BM				...									
B	Sa	BS, BM					...								
B	Su	BS, BM						...							
G	Mo	GM, BM							...						
G	Tu	GT, BM								...					
G	We	GW, BM									...				
G	Th	GT, BM										...			
G	Fr	GF, BM											...		
G	Sa	GS,												...	

S. Siriwardena
ss2032@cam.ac.uk

		BM													
G	Su	GS, BM													...

The LIGHT BLUE immediately above indicates all the possible pairs of children where one is a girl born on a Monday. There are **27** such pairs.

Now all that's left to do is sort out how many of *these* pairs are pairs of children where BOTH children are girls. I.e. how many events are in DARK BLUE below. Answer: **13**

		BOY							GIRL						
		Mo	Tu	We	Th	Fr	Sa	Su	Mo	Tu	We	Th	Fr	Sa	Su
B	Mo	BM, BM	BM, BT	BM, BW	BM, BT	BM, BF	BM, BS	BM, BS	BM, GM	BM, GT	BM, GW	BM, GT	BM, GF	BM, GS	BM, GS
B	Tu	BT, BM	...												
B	We	BW, BM		...											
B	Th	BT, BM			...										
B	Fr	BF, BM				...									
B	Sa	BS, BM					...								
B	Su	BS, BM						...							
G	Mo	GM, BM							...						
G	Tu	GT, BM								...					
G	We	GW, BM									...				
G	Th	GT, BM										...			
G	Fr	GF, BM											...		
G	Sa	GS, BM												...	

S. Siriwardena
ss2032@cam.ac.uk

G	Su	GS, BM													...

THUS, the probability that Jill has two girls given that she has one girl born on a Monday is **13/27**.