Shyane Siriwardena
ss2032@cam.ac.uk

A Guide to FOL Proof Rules

(for Worksheet 6)

This lesson sheet will be a good deal like last class's. This time, I'll be running through the proof rules relevant to FOL. Of course, when you're doing any given proof, these are used *in addition to* the TFL rules. If you need a refresher on those, have a look at the last handout as well.

Since FOL works with interpretations rather than truth-tables, the explanations will be somewhat different than many of those you saw last time. However, the objective here will be the same: to help the rules make sense with respect to what you already know about the quantifiers.

Universal Elimination (∀E)

This rule is probably the simplest of all of those unique to FOL. Suppose we had a proposition $\forall x Fx$. This proposition tells us that *everything* is F. So, we can eliminate the quantifier by saying of *any object* that it is F: e.g. *Fa, Fb, Fc, etc..* With this rule, it does not matter whether the name already appears in the proof of not. If everything is F, then that includes the objects already named in the proof!

Another thing to remember with ∀E -- you must replace *all* instances of the relevant variable in the proposition. What do I mean by this?

Suppose we had a proposition that looked like this:

$$\forall x\, (Fx \rightarrow ((Qxa \vee Rbx) \longleftrightarrow Sc)$$

If you were to use the ∀E rule on this proposition, you would have to replace *every* instance of the variable under the scope of the variable--i.e. you would have to replace every instance of *x* in this case.

So, the proposition you derive might look like this

$(Fa \rightarrow ((Qaa \vee Rba) \longleftrightarrow Sc)$ ✓ or $(Fe \rightarrow ((Qea \vee Rbe) \longleftrightarrow Sc)$ ✓

but NOT like this

$(Fa \rightarrow ((Qxa \vee Rbx) \longleftrightarrow Sc)$ ✗ or $(Fa \rightarrow ((Qba \vee Rbb) \longleftrightarrow Sc)$ ✗

Shyane Siriwardena
ss2032@cam.ac.uk

Here, then, is the formal statement of the $\forall$E rule:

m	$\forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots)$	
	$\mathcal{A}(\ldots c \ldots c \ldots)$	$\forall$E m

Existential Introduction (**$\exists$I**)

This rule allows us to move from saying of some particular object *a* that it is F, to saying that *something* is F. For any given proposition that has a name in it, we can replace *any* of the instances of that name by way of $\exists$I. In this way, $\exists$I differs from universal elimination ($\forall$E). Observe, suppose we had a proposition that looked like this:

$$\forall x(As \rightarrow ((Bsx \vee Cxs) \longleftrightarrow Dc)$$

And suppose that we wanted to replace the name 's'. In this case, we can replace as many or as few of the instances of 's' as we like. The only restriction is, I couldn't use a variable that already appears in the proposition. In other words, I could not do this:

$$\exists x \forall x(Ax \rightarrow ((Bxx \vee Cxx) \longleftrightarrow Dc) \text{ ✗}$$

We already had a quantifier quantifying over x's. In doing this, there's now conflict between the two quantifiers. This is never allowed. Instead, we would have to do this:

$$\exists y \forall x(Ay \rightarrow ((Byx \vee Cxy) \longleftrightarrow Dc) \text{ ✓}$$

Similarly, either of the following would also be acceptable, since I don't have to replace all instances of *s*:

$\exists y \forall x(As \rightarrow ((Byx \vee Cxy) \longleftrightarrow Dc)$ ✓ OR $\exists y \forall x(Ay \rightarrow ((Bsx \vee Cxs) \longleftrightarrow Dc)$ ✓

Here, then, is the formal rule:

m	$\mathcal{A}(\ldots c \ldots c \ldots)$	
	$\exists \chi \mathcal{A}(\ldots \chi \ldots c \ldots)$	$\exists$I m

χ must not occur in $\mathcal{A}(\ldots c \ldots c \ldots)$

Shyane Siriwardena
ss2032@cam.ac.uk

A brief (but important!) note on the placement of quantifiers: we *always* build our proposition from the inside out. So, when you introduce a quantifier it <u>must</u> be placed furthest to the left, relative to any quantifiers that are already there. The quantifier you introduce has to be the <u>outermost</u> in the proposition at that time. For instance, you couldn't move from this

$$\forall x(As \rightarrow ((Bsx \vee Cxs) \longleftrightarrow Dc)$$

to...

$$\forall x \exists y(Ay \rightarrow ((Byx \vee Cxy) \longleftrightarrow Dc) \text{ ✗}$$

The existential quantifier in this case *has* to go outside the quantifier that was there first, as in the following:

$$\exists y \forall x(Ay \rightarrow ((Byx \vee Cxy) \longleftrightarrow Dc) \text{ ✓}$$

This means that, when there are several quantifiers in the conclusion you're trying to prove, you need to mindful of the order in which they appear. You might find you have to introduce the quantifiers in a particular order so that they appear in the order called for.

In general, whenever you introduce a connective or quantifier, it must be the main operator in the resulting proposition. Similarly, whenever you eliminate a connective or quantifier, it must be the main operator in the proposition containing that connective or quantifier.

Universal Introduction (∀I)

Universal quantifiers range over *everything* in our domain. This makes proving a universal tricky! It's not enough to show that one or two things are F in order to prove that $\forall xFx$ -- we need to show that *everything* is F. Obviously, we couldn't do this manually, else we'd never finish the proof! (After all, some domains have infinitely many members.) So instead, we take a slightly different approach. We show that, regardless of which element of the domain you picked, that thing would be F. Indeed, to do this, we don't even have to know how many things there are in the domain.

Imagine that I told you "Every person has a heart and a liver. So, everyone has a heart." And imagine someone (who clearly wasn't paying attention in logic class!) responded, "Okay, I believe everyone has a heart and a liver, but how do you know they all have hearts?". I could prove without having to check any person at all. "Take any person you like. Bob, for instance," I'd say. "Since we both accept that every person has a heart and a liver, Bob has a heart and a liver. But, it follows from that that Bob has a heart." Now, you might worry that I've only said something about Bob--not every person. But I said to pick anyone you like--*nothing hangs on being Bob*. The key is, the reasoning would have worked no matter who I picked. Watch:

Shyane Siriwardena
ss2032@cam.ac.uk

1.	Every person has a heart and a liver.	PREMISE

Now pick a random person: Chuck

2.	Chuck has a heart and a liver.	UNIVERSAL ELIMINATION, 1
3.	Chuck has a heart	CONJUNCTION ELIMINATION, 2

But I could've picked anyone other than Chuck. In my reasoning, I didn't have to use any facts unique to Chuck. None of my premises or assumptions said anything about Chuck in particular. So, the exact same reasoning would have worked for Donna through Zoe, etc.

∴	Every person has a heart	UNIVERSAL INTRODUCTION, 3

In a proper proof, when we apply this rule, the name that we're replacing *cannot appear in any undischarged assumptions*. An assumption is undischarged, recall, when we're still inside of its scope -- visually, when you're still working to the right of that vertical line. So, (and this is easy to forget) premises count as a kind of assumption, and we are always in the scope of that assumption, since premises are out on the left-most scope-line -- the one we called the "reality" line. That means, if a name shows up in one of the premises, *you cannot apply* $\forall\mathrm{I}$ *to that name!* If we could, we would be allowed to make the following obviously fallacious argument:

1.	Chuck is a poet.	PREMISE
∴	Everyone is a poet.	UNIVERSAL INTRODUCTION, 1

Here, then, is the formal statement of the $\forall\mathrm{I}$ rule:

$$
\begin{array}{r|ll}
m & \mathcal{A}(\ldots c \ldots c \ldots) & \\
 & \forall \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots) & \forall\mathrm{I}\ m
\end{array}
$$

c must not occur in any undischarged assumption
χ must not occur in $\mathcal{A}(\ldots c \ldots c \ldots)$

(Recall from the first class this term, I mentioned that this method of reasoning about an arbitrary element of the domain would return. We first saw this in the "Reasoning about All Interpretations" section. That section was essentially describing (in prose rather than formal language) a kind of universal introduction.)

Shyane Siriwardena
ss2032@cam.ac.uk

Existential Elimination (∃E)

The final basic quantifier rule is Existential Elimination. This rule is a bit unusual in the way that the rule for disjunction elimination was unusual, since we aren't eliminating the quantifier in the way that we eliminate other operators. When we have some existential proposition ∃*xFx*, we know that *something* is F, but we don't know which thing. So, to eliminate an existential, we suppose that some arbitrary thing is F, and see what follows. For instance, consider the following:

Suppose I told you, "Someone is from Canada. And for anyone, if they're from Canada, then they're from North America. So, someone is from North America."

But, we don't know who's from Canada, so how do we show this is valid?

Well, take someone--anyone you like!--Harry, let's say. Let's *suppose* Harry's from Canada. Since, if a person's from Canada, then they're from North America, Harry is also from North America.

What can I conclude from this reasoning?

NOT that Harry is from North America! Harry was a random person we picked. We don't know anything about Harry. Everything we said about Harry was *in the scope of the assumption that he is from Canada*. And, once I've picked him for our assumption, I cannot take Harry's name outside of that assumption.

Here's where we've gotten so far:

1.	Someone is from Canada.	PREMISE
2.	For all people, if they're from Canada, then they're from N. America.	PREMISE
3.	Harry is from Canada	ASSUMPTION
4.	If Harry is from Canada, then Harry is from N. America.	∀E 2
5.	Harry is from N. America	→E 4, 3

Now, we just said I can't take 'Harry' out of the scope of the assumption, out to the reality line. So I need to derive a proposition without 'Harry' before I can discharge the assumption. Of course, if Harry is from North America, it follows that *someone* is from North America. (Remember, whenever we have a name, we can introduce an existential.) So:

1.	Someone is from Canada.	PREMISE
2.	For all people, if they're from Canada, then they're from N. America.	PREMISE
3.	Harry is from Canada	ASSUMPTION
4.	If Harry is from Canada, then Harry is from N. America.	∀E 2
5.	Harry is from N. America	→E 4, 3
6.	**Someone is from N. America**	**∃I 5**

Shyane Siriwardena
ss2032@cam.ac.uk

Now we can discharge the assumption. Line 6 doesn't say anything about Harry anymore, so we're allowed to conclude the argument as follows:

1.	Someone is from Canada.	PREMISE
2.	For all people, if they're from Canada, then they're from N. America.	PREMISE
3.	Harry is from Canada	ASSUMPTION
4.	If Harry is from Canada, then Harry is from N. America.	∀E 2
5.	Harry is from N. America	→E 4, 3
6.	Someone is from N. America	∃I 5
7.	**Someone is from N. America**	**EXISTENTIAL ELIMINATION 1, 3-6**

Let's make sense of that rule citation. Why did I cite Line 1? Answer: because that is the existential statement that we used to generate our assumption -- that is the existential we are 'eliminating'. After that, we have to cite the *entire* sub-proof, since that is how we got to the proposition in Line 7.

Once again, when we decided to use 'Harry' in our assumption in Line 3, it's important that 'Harry' didn't show up anywhere else in our argument (i.e. Harry didn't appear in any undischarged assumptions). 'Harry', in other words, was completely arbitrary. Exactly the same reasoning would have worked if we had used 'Ingrid' or 'Jess'.

Here is the formal statement of the rule for ∃E:

$$\begin{array}{r|l l}
m & \exists \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots) & \\
i & \quad \mathcal{A}(\ldots c \ldots c \ldots) & \\
j & \quad \mathcal{B} & \\
 & \mathcal{B} & \exists\mathrm{E}\ m,\ i\text{–}j
\end{array}$$

c must not occur in any assumption undischarged before line i
c must not occur in $\exists \chi \mathcal{A}(\ldots \chi \ldots \chi \ldots)$
c must not occur in $\mathcal{B}$

That exhausts the basic quantifier rules. Beyond these, there is a collection of derived rules called "conversion of quantifiers" which we'll turn to next.

Shyane Siriwardena
ss2032@cam.ac.uk

Conversion of Quantifiers (CQ)

The CQ rules allow us to move negations around our quantifiers more easily. Thinking about these rules in natural(-ish) language, they become quite intuitive. Let's look at an example using the property 'is a walrus'.

- Suppose **<u>not</u> everything** is a walrus...
  ...then, it *must* be the case that **there exists** a **<u>not</u>**-walrus (i.e. a thing that's not a walrus)

HENCE:

m	$\neg\forall\chi\mathcal{A}$	
	$\exists\chi\neg\mathcal{A}$	CQ m

AND VICE VERSA...

- Suppose **there exists** a **<u>not</u>**-walrus (i.e. a thing that's not a walrus)...
  ...then, it *must* be the case that **<u>not</u> everything** is a walrus

HENCE:

m	$\exists\chi\neg\mathcal{A}$	
	$\neg\forall\chi\mathcal{A}$	CQ m

- Suppose **everything** is **<u>not</u>** a walrus…
  ...then, it must be the case that **<u>no</u>thing exists** that is a walrus.

HENCE:

m	$\forall\chi\neg\mathcal{A}$	
	$\neg\exists\chi\mathcal{A}$	CQ m

Shyane Siriwardena
ss2032@cam.ac.uk

AND VICE VERSA...

- Suppose **__no__thing exists** that is a walrus…
  ...then, it must be the case that **everything** is **__not__** a walrus

  HENCE:

m	$\neg\exists\chi\mathcal{A}$	
	$\forall\chi\neg\mathcal{A}$	CQ m

In general, it’s useful to keep the following in mind:

Whenever you move a negation “through” a quantifier...
(i.e. if you push or pull it to the opposite side of the quantifier)

↓

...Flip the quantifier
(i.e. change the quantifier to its opposite)

Bear in mind(!), this only works when the negation is *immediately adjacent to* the quantifier in question. And, by way of this rule, that negation can only move to a place that is also immediately adjacent to the quantifier.

IDENTITY

The identity rules are relatively straightforward. At any point in a proof, and for any name *c*, you can always say *c* = *c*. This is identity introduction.

$c = c$	$=$I

As for identity elimination, whenever you have some identity claim *a* = *b*, then whenever you have an instance of *either one of those names*, you can replace it with the other. And if the name shows up more than once in a given proposition, you can replace *as many or as few* instances as you like.

Shyane Siriwardena
ss2032@cam.ac.uk

Hence:

$$\begin{array}{r|ll} m & a = b & \\ n & \mathcal{A}(\ldots a \ldots a \ldots) & \\ & \mathcal{A}(\ldots b \ldots a \ldots) & =\text{E } m, n \end{array}$$

And

$$\begin{array}{r|ll} m & a = b & \\ n & \mathcal{A}(\ldots b \ldots b \ldots) & \\ & \mathcal{A}(\ldots a \ldots b \ldots) & =\text{E } m, n \end{array}$$

The trickier part about these rules is knowing just *when* to use them! Particularly =I. This will come with practice. This is one of those rules that is easy to forget, so whenever you get stuck on proof involving identity, remember to try *both* identity rules to see what they can get you.